Chapter - Determinants

Topic-1: Minor & Co-factor of an Element of a Determinant, Value of a Determinant

1 MCQs with One Correct Answer

1. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is [2011]
   (a) 2 (b) 6 (c) 4 (d) 8

2. Consider three points
   $P = (-\sin(\beta - \alpha), -\cos\beta)$, $Q = (\cos(\beta - \alpha), \sin\beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \alpha, \beta, \theta < \frac{\pi}{4}$.
   Then, [2008]
   (a) P lies on the line segment RQ
   (b) Q lies on the line segment PR
   (c) R lies on the line segment QP
   (d) P, Q, R are non-collinear

3. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1. Let C be the subset of A consisting of all determinants with value –1. Then [1981 - 2 Marks]
   (a) C is empty
   (b) B has as many elements as C
   (c) $A = B \cup C$
   (d) B has twice as many elements as elements as C

2 Integer Value Answer/ Non-Negative Integer

4. Let $S = \left\{ A = \begin{pmatrix} 0 & 1 & c \\ 1 & a & d \\ 1 & b & e \end{pmatrix} : a, b, c, d, e \in \{0,1\} \text{ and } |A| \in \{-1, 1\} \right\}$ where $|A|$ denotes the determinant of A. Then the number of elements in S is ______. [Adv. 2024]

5. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2×2 matrix such that the trace of A is 3 and the trace of A^3 is -18, then the value of the determinant of A is ______ [Adv. 2020]

6. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is ____. [Adv. 2018]

7. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s $\in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is [Adv. 2016]

4 Fill in the Blanks

8. The solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is .... [1981 - 2 Marks]

9. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$ be an identity in λ, where p, q, r, s and t are constants. Then, the value of t is .... [1981 - 2 Marks]

6 MCQs with One or More than One Correct Answer

10. Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then [Adv. 2014]
(a) determinant of $(M^2 + MN^2)$ is 0
(b) there is 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
(c) determinant of $(M^2 + MN^2) \geq 1$
(d) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

7 Match the Following

11. Consider the lines given by
$L_1 : x + 3y - 5 = 0; L_2 : 3x - ky - 1 = 0; L_3 : 5x + 2y - 12 = 0$
Match the Statements / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS. [2008]

Column I	Column II
(A) L_1, L_2, L_3 are concurrent, if	(p) $k = -9$
(B) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q) $k = -\frac{6}{5}$
(C) L_1, L_2, L_3 from a triangle, if	(r) $k = \frac{5}{6}$
(D) L_1, L_2, L_3 do not form a triangle, if	(s) $k = 5$

8 Comprehension/Passage Based Questions

PASSAGE

Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_P = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, ..., p-1\} \right\}$$ [2010]

12. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det(A) divisible by p is
(a) $(p-1)^2$ (b) $2(p-1)$
(c) $(p-1)^2 + 1$ (d) $2p - 1$

13. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is
[**Note:** The trace of a matrix is the sum of its diagonal entries.]
(a) $(p-1)(p^2 - p + 1)$ (b) $p^3 - (p-1)^2$
(c) $(p-1)^2$ (d) $(p-1)(p^2-2)$

14. The number of A in T_p such that det (A) is not divisible by p is
(a) $2p^2$ (b) $p^3 - 5p$
(c) $p^3 - 3p$ (d) $p^3 - p^2$

10 Subjective Problems

15. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.
[1981 - 4 Marks]

Topic-2: Properties of Determinants, Area of a Triangle

1 MCQs with One Correct Answer

1. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \leq i, j \leq 3$. If the determinant of P is 2, then the determinant of the matrix Q is [2012]
(a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

2. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$ then the value of α is [2004S]
(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5

3. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant
$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is [2002S]
(a) 3ω (b) $3\omega(\omega - 1)$ (c) $3\omega^2$ (d) $3\omega(1 - \omega)$

4. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$ then $f(100)$ is equal to [1999 - 2 Marks]
(a) 0 (b) 1 (c) 100 (d) –100

5. The parameter, on which the value of the determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend upon is [1997 - 2 Marks]
(a) a (b) p (c) d (d) x

6. If $\omega (\neq 1)$ is a cube root of unity, then
$\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2 - 1 \\ -i & -i+\omega-1 & -1 \end{vmatrix} =$ [1995S]
(a) 0 (b) 1 (c) i (d) ω

2 Integer Value Answer/ Non-Negative Integer

7. Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$ is equal to [2010]

4 Fill in the Blanks

8. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is................ [1993 - 2 Marks]

9. The value of the determinant

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$$ is................ [1988 - 2 Marks]

10. Given that $x=-9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}=0$ the other two roots are .... and .... [1983 - 2 Marks]

5 True / False

11. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$ then the two triangles with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3), and (a_1,b_1), (a_2,b_2), (a_3,b_3) must be congruent. [1985 - 1 Mark]

12. The determinants

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \text{ and } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ are not identically equal. [1983 - 1 Mark]

6 MCQs with One or More than One Correct Answer

13. Let $|M|$ denote the determinant of a square matrix M. Let $g:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}$ be the function defined by

where $g(\theta)=\sqrt{f(\theta)-1}+\sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$

$$f(\theta)=\frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}+\begin{vmatrix} \sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right) \\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right) \\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi \end{vmatrix}.$$

Let $p(x)$ be a quadratic polynomial whose roots are the maximum and minimum values of the function $g(\theta)$, and $p(2)=2-\sqrt{2}$. Then, which of the following is/are TRUE ? [Adv. 2022]

(a) $p\left(\frac{3+\sqrt{2}}{4}\right)<0$ (b) $p\left(\frac{1+3\sqrt{2}}{4}\right)>0$

(c) $p\left(\frac{5\sqrt{2}-1}{4}\right)>0$ (d) $p\left(\frac{5-\sqrt{2}}{4}\right)<0$

14. Which of the following is(are) not the square of a 3 × 3 matrix with real entries? [Adv. 2017]

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

15. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix}=-648\alpha?$$

[Adv. 2015]

(a) –4 (b) 9 (c) –9 (d) 4

16. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}=x+iy$, then [1998 - 2 Marks]

(a) $x=3, y=1$ (b) $x=1, y=3$
(c) $x=0, y=3$ (d) $x=0, y=0$

17. The values of θ lying between θ = 0 and θ = π/2 and satisfying the equation [1988 - 2 Marks]

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix}=0$$ are

(a) $7\pi/24$ (b) $5\pi/24$ (c) $11\pi/24$ (d) $\pi/24$.

7 Match the Following

18. Let α and β be the distinct roots of the equation $x^2 + x - 1 = 0$. Consider the set $T = \{1, \alpha, \beta\}$ For a 3×3 matrix $M = (a_{ij})_{3\times3}$, define $R_i = a_{i1} + a_{i2} + a_{i3}$ and $C_j = a_{1j} + a_{2j} + a_{3j}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. Match each entry in List-I to the correct entry in List-II.

List-I	List-II
(P) The number of matrices $M = (a_{ij})_{3\times3}$ with all entries in T such that $R_i = C_j = 0$ for all i, j is	(1) 1
(Q) The number of symmetric matrices $M = (a_{ij})_{3\times3}$ with all entries in T such that $C_j = 0$ for all j, is	(2) 12
(R) Let $M = (a_{ij})_{3\times3}$ be a skew symmetric matrix such that $a_{ij} \in T$ for $i > j$. Then the number of elements in the set $\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R}, M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{12} \\ 0 \\ -a_{23} \end{pmatrix}\right\}$ is	(3) infinite
(S) Let $M = (a_{ij})_{3\times3}$ be a matrix with all entries in T such that $R_i = 0$ for all i. Then the absolute value of the determinant of M is	(4) 6
	(5) 0

The correct option is **[Adv. 2024]**

(a) (P)→(4) (Q)→(2) (R)→(5) (S)→(1)
(b) (P)→(2) (Q)→(4) (R)→(1) (S)→(5)
(c) (P)→(2) (Q)→(4) (R)→(3) (S)→(5)
(d) (P)→(1) (Q)→(5) (R)→(3) (S)→(4)

10 Subjective Problems

19. If M is a 3×3 matrix, where $\det M = 1$ and $MM^T = I$, where 'I' is an identity matrix, prove that $\det(M - I) = 0$. **[2004 - 2 Marks]**

20. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^T A = I$, then find the value of $a^3 + b^3 + c^3$. **[2003 - 2 Marks]**

21. Prove that for all values of θ,

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$$

[2000 - 3 Marks]

22. Let $a > 0$, $d > 0$. Find the value of the determinant **[1996 - 5 Marks]**

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

23. For all values of A, B, C and P, Q, R show that **[1994 - 4 Marks]**

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

24. For a fixed positive integer n, if **[1992 - 4 Marks]**

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$ then show that

$\left[\frac{D}{(n!)^3} - 4\right]$ is divisible by n.

25. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$. Then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$ **[1991 - 4 Marks]**

26. Let the three digit numbers A 28, $3B9$, and $62C$, where A, B, and C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k. **[1990 - 4 Marks]**

27. Let $\Delta a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$. Show that $\sum_{a=1}^{n} \Delta a = c$, a constant. **[1989 - 5 Marks]**

28. Show that

$$\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

[1985 - 2 Marks]

29. Without expanding a determinant at any stage, show that

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$$, where A and B are determinants of order 3 not involving x. [1982 - 5 Marks]

Topic-3: Adjoint of a Matrix, Inverse of a Matrix, Some Special Cases of Matrix, Rank of a Matrix

1 MCQs with One Correct Answer

1. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if [Adv. 2014]
   (a) The first column of M is the transpose of the second row of M
   (b) The second row of M is the transpose of the first column of M
   (c) M is a diagonal matrix with non-zero entries in the main diagonal
   (d) The product of entries in the main diagonal of M is not the square of an integer

2. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and [2005S]

   $A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$, then the value of c and d are

   (a) $(-6, -11)$ (b) $(6, 11)$ (c) $(-6, 11)$ (d) $(6, -11)$

2 Integer Value Answer/ Non-Negative Integer

3. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$$

If det (adj A) + det (adj B) = 10^6, then [k] is equal to

[**Note** : adj M denotes the adjoint of square matrix M and $[k]$ denotes the largest integer less than or equal k. [2010]

6 MCQs with One or More than One Correct Answer

4. For any 3×3 matrix M, let $|M|$ denote the determinant of M. Let

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a non-singular matrix of order 3×3, then which of the following statements is (are) TRUE ? [Adv. 2021]

   (a) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   (b) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$
   (c) $|(EF)^3| > |EF|^2$
   (d) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

5. For any 3×3 matrix M, let $|M|$ denote the determinant of M. Let I be the 3×3 identity matrix. Let E and F be two 3×3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) TRUE? [Adv. 2021]
   (a) $|FE| = |I - FE||FGE|$ (b) $(I - FE)(I + FGE) = I$
   (c) $EFG = GEF$ (d) $(I - FE)(I - FGE) = I$

6. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If M^{-1} = adj (adj M), then which of the following statements is/are ALWAYS TRUE?
   (a) $M = I$ (b) det $M = 1$ [Adv. 2020]
   (c) $M^2 = I$ (d) $(\text{adj } M)^2 = I$

7. Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

$$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and $X = \sum_{k=1}^{6} P_k \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_k^T$

Where P_k^T denotes the transpose of the matrix P_k. Then which of the following options is/are correct? [Adv. 2019]
   (a) X is a symmetric matrix
   (b) The sum of diagonal entries of X is 18
   (c) $X - 30I$ is an invertible matrix
   (d) If $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then a = 30

8. Let $M=\begin{bmatrix}0&1&a\\1&2&3\\3&b&1\end{bmatrix}$ and $(\text{adj } M)=\begin{bmatrix}-1&1&-1\\8&-6&2\\-5&3&-1\end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ? [Adv. 2019]
(a) $a+b=3$
(b) $\det(\text{adj } M^2)=81$
(c) $(adjM)^{-1}+adjM^{-1}=-M$
(d) $If\ M\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$, then $\alpha-\beta+\gamma=3$

9. Let $x\in R$ and let $P=\begin{bmatrix}1&1&1\\0&2&2\\0&0&3\end{bmatrix}, Q=\begin{bmatrix}2&x&x\\0&4&0\\x&x&6\end{bmatrix}$ and $R=PQP^{-1}$
Then which of the following options is/are correct? [Adv. 2019]
(a) $\det R=\det\begin{bmatrix}2&x&x\\0&4&0\\x&x&5\end{bmatrix}+8$, for all $x\in R$
(b) For $x=1$, there exists a unit vector $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ for which $R\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$
(c) There exists a real number x such that $PQ=QP$
(d) For $x=0$, if $R\begin{bmatrix}1\\a\\b\end{bmatrix}=6\begin{bmatrix}1\\a\\b\end{bmatrix}$, then $a+b=5$

10. For 3×3 matrices M and N, which of the following statement(s) is (are) NOT correct? [Adv. 2013]
(a) N^TMN is symmetric or skew symmetric, according as M is symmetric or skew symmetric
(b) MN – NM is skew symmetric for all symmetric matrices M and N
(c) MN is symmetric for all symmetric matrices M and N
(d) (adj M) (adj N) = adj (MN) for all invertible matrices M and N

11. Let M and N be two 3×3 non-singular skew- symmetric matrices such that MN = NM. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to [2011]
(a) M^2 (b) $-N^2$ (c) $-M^2$ (d) MN

8 Comprehension/Passage Based Questions

PASSAGE

Let $A=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$, and U_1, U_2 and U_3 are columns of a 3×3 matrix U. If column matrices U_1, U_2 and U_3 satisfying $AU_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, AU_2=\begin{bmatrix}2\\3\\0\end{bmatrix}, AU_3=\begin{bmatrix}2\\3\\1\end{bmatrix}$ evaluate as directed in the following questions. [2006 - 5M, –2]

12. The value $|U|$ is
(a) 3 (b) –3 (c) $\frac{3}{2}$ (d) 2

13. The sum of the elements of the matrix U^{-1} is
(a) –1 (b) 0 (c) 1 (d) 3

14. The value of $[3\ 2\ 0]\,U\begin{bmatrix}3\\2\\0\end{bmatrix}$ is
(a) 5 (b) $\frac{5}{2}$ (c) 4 (d) $\frac{3}{2}$

Topic-4: Solution of System of Linear Equations

1 MCQs with One Correct Answer

1. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ has exactly two distinct solutions, is [2010]
(a) 0 (b) 2^9-1 (c) 168 (d) 2

2. Given $2x-y+2z=2, x-2y+z=-4, x+y+\lambda z=4$ then the value of λ such that the given system of equation has NO solution, is [2004S]
(a) 3 (b) 1 (c) 0 (d) –3

3. If the system of equations $x+ay=0$, $az+y=0$ and $ax+z=0$ has infinite solutions, then the value of a is [2003S]
(a) –1 (b) 1
(c) 0 (d) no real values

4. The number of values of k for which the system of equations $(k+1)x+8y=4k;\ kx+(k+3)y=3k-1$ has infinitely many solutions is [2002S]
(a) 0 (b) 1 (c) 2 (d) infinite

5. If the system of equations
$x-ky-z=0, kx-y-z=0, x+y-z=0$ has a non-zero solution, then the possible values of k are [2000S]
(a) $-1, 2$ (b) $1, 2$ (c) $0, 1$ (d) $-1, 1$

6. Let a, b, c be the real numbers. Then following system of equations in x, y and z [1995S]
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$
$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ has
(a) no solution (b) unique solution
(c) infinitely many solutions (d) finitely many solutions

2 Integer Value Answer/ Non-Negative Integer

7. For a real number α, if the system
$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ of linear equations, has infinitely many solutions, then $1+\alpha+a^2=$ [Adv. 2017]

3 Numeric / New Stem Based Questions

Question Stem for Question Nos. 8 and 9

Let α, β and γ be real numbers such that the system of linear equations
$x+2y+3z=\alpha$
$4x+5y+6z=\beta$
$7x+8y+9z=\gamma-1$
is consistent. Let $|M|$ represent the determinant of the matrix

$$M=\begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point (0, 1, 0) from the plane P.

8. The value of $|M|$ is _____. [Adv. 2021]
9. The value of D is _____. [Adv. 2021]

4 Fill in the Blanks

10. The system of equations
$\lambda x+y+z=0$
$-x+\lambda y+z=0$
$-x-y+\lambda z=0$
Will have a non-zero solution if real values of λ are given by.................. [1984 - 2 Marks]

7 Match the Following

11. Let p, q, r be nonzero real numbers that are, respectively, the 10[th], 100[th] and 1000[th] terms of a harmonic progression. Consider the system of linear equations
$x+y+z=1$
$10x+100y+1000z=0$
$qr\,x+pr\,y+pq\,z=0.$ [Adv. 2022]

List-I	List-II
(I) If $\frac{q}{r}=10$, then the system of linear equations has	(P) $x=0, y=\frac{10}{9}, z=-\frac{1}{9}$ as a solution
(II) If $\frac{p}{r}\neq 100$, then the system of linear equations has	(Q) $x=\frac{10}{9}, y=-\frac{1}{9}, z=0$ solution
(III) If $\frac{p}{q}\neq 10$, then the system of linear equations has	(R) infinitely many solutions
(IV) If $\frac{p}{q}=10$, then the system of linear equations has	(S) no solution
	(T) at least one solution

The correct option is:
(a) (I)→(T); (II)→(R); (III)→(S); (IV)→(T)
(b) (I)→(Q); (II)→(S); (III)→(S); (IV)→(R)
(c) (I)→(Q); (II)→(R); (III)→(P); (IV)→(R)
(d) (I)→(T); (II)→(S); (III)→(P); (IV)→(T)

6 MCQs with One or More than One Correct Answer

12. Let $M=(a_{ij})$, $i, j \in \{1,2,3\}$, be the 3×3 matrix such that $a_{ij}=1$ if $j+1$ is divisible by i, otherwise $a_{ij}=0$. Then which of the following statements is(are) true? [Adv. 2023]
(a) M is invertible.
(b) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $M\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}=\begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$.
(c) The set $\{X\in\mathbb{R}^3 : MX=0\}\neq\{0\}$ where $0=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
(d) The matrix $(M-2I)$ is invertible, where I is the 3×3 identity matrix.

13. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3, \in \mathbb{R}$ and the system of equations (in real variables)

$$-x+2y+5z=b_1$$
$$2x-4y+3z=b_2$$
$$x-2y+2z=b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? [Adv. 2018]

(a) $x+2y+3z=b_1, 4y+5z=b_2$ and $x+2y+6z=b_3$
(b) $x+y+3z=b_1, 5x+2y+6z=b_2$ and $-2x-y-3z=b_3$
(c) $-x+2y-5z=b_1, 2x-4y+10z=b_2$ and $x-2y+5z=b_3$
(d) $sx+2y+5z=b_1, 2x+3z=b_2$ and $x+4y-5z=b_3$

14. Let $a, \lambda, \mu, \in \mathbb{R}$. Consider the system of linear equations

$$ax+2y=\lambda$$
$$3x-2y=\mu$$

Which of the following statement(s) is (are) correct? [Adv. 2016]

(a) If $a=-3$, then the system has infinitely many solutions for all values of λ and μ.
(b) If $a\neq-3$, then the system has a unique solution for all values of λ and μ.
(c) If $\lambda+\mu=0$, then the system has infinitely many solutions for $a=-3$.
(d) If $\lambda+\mu\neq0$, then the system has no solution for $a=-3$.

7 Match the Following

15. Let α, β and γ be real numbers. Consider the following system of linear equations [Adv. 2023]

$$x+2y+z=7$$
$$x+\alpha z=11$$
$$2x-3y+\beta z=\gamma$$

Match each entry in List-I to the correct entries in List-II.

List-I	List-II
(P) If $\beta=\frac{1}{2}(7\alpha-3)$ and $\gamma=28$, then the system has	(1) a unique solution
(Q) If $\beta=\frac{1}{2}(7\alpha-3)$ and $\gamma\neq28$, then the system has	(2) no solution
(R) If $\beta\neq\frac{1}{2}(7\alpha-3)$ where $\alpha=1$ and $\gamma\neq28$, then the system has	(3) infinitely many solutions
(S) If $\beta\neq\frac{1}{2}(7\alpha-3)$ where $\alpha=1$ and $\gamma=28$, then the system has	(4) $x=11, y=-2$ and $z=0$ as a solution
	(5) $x=-15, y=4$ and $z=0$ as a solution

The correct option is:

(a) (P)→(3), (Q)→(2), (R)→(1), (S)→(4)
(b) (P)→(3), (Q)→(2), (R)→(5), (S)→(4)
(c) (P)→(2), (Q)→(1), (R)→(4), (S)→(5)
(d) (P)→(2), (Q)→(1), (R)→(1), (S)→(3)

8 Comprehension/Passage Based Questions

PASSAGE

Let $\mathscr{A}$ be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. [2009]

16. The number of matrices in $\mathscr{A}$ is

(a) 12 (b) 6 (c) 9 (d) 3

17. The number of matrices A in $\mathscr{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution, is

(a) less than 4
(b) at least 4 but less than 7
(c) at least 7 but less than 10
(d) at least 10

18. The number of matrices A in $\mathscr{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is inconsistent, is

(a) 0 (b) more than 2
(c) 2 (d) 1

9 Assertion and Reason/Statement Type Questions

19. Consider the system of equations

$$x-2y+3z=-1$$
$$-x+y-2z=k$$
$$x-3y+4z=1$$

STATEMENT - 1 : The system of equations has no solution for $k\neq3$ and

STATEMENT-2 : The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$. [2008]

(a) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1

(b) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is NOT a correct explaination for STATEMENT - 1

(c) STATEMENT - 1 is True, STATEMENT - 2 is False

(d) STATEMENT - 1 is False, STATEMENT - 2 is True

10 Subjective Problems

20. If $A = \begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}, B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $AX = U$ has infinitely many solutions, prove that $BX = V$ has no unique solution. Also show that if $afd \neq 0$, then $BX = V$ has no solution. [2004 - 4 Marks]

21. Let λ and α be real. Find the set of all values of λ for which the system of linear equations [1993 - 5 Marks]

$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0,$

$x + (\cos\alpha)y + (\sin\alpha)z = 0,$

$-x + (\sin\alpha)y - (\cos\alpha)z = 0$

has a non-trivial solution. For $\lambda = 1$, find all values of α.

22. Consider the system of linear equations in x, y, z:

$(\sin 3\theta)\ x - y + z = 0$

$(\cos 2\theta)\ x + 4y + 3z = 0$

$2x + 7y + 7z = 0$

Find the values of θ for which this system has nontrivial solutions. [1986 - 5 Marks]

23. For what value of k do the following system of equations possess a non trivial (i.e., not all zero) solution over the set of rationals Q?

$x + ky + 3z = 0$

$3x + ky - 2z = 0$

$2x + 3y - 4z = 0$

For that value of k, find all the solutions for the system. [1979]

Answer Key

Topic-1 : Minor & Co-factor of an Element of a Determinant, Value of a Determinant

1. (a)	2. (d)	3. (b)	4. (16)	5. (5)	6. (4)	7. (1)	8. (–1, 2)	9. (0)	10. (a, b)
11. (A)-s; (B)-p, q; (C)-r; (D)-p, q, s	12. (d)	13. (c)	14. (d)						

Topic-2 : Properties of Determinants, Area of a Triangle

1. (d)	2. (c)	3. (b)	4. (a)	5. (b)	6. (a)	7. (0)	8. (0)	9. (0)	10. (2, 7)
11. (False)	12. (False)	13. (a,c)	14. (b,d)	15. (b,c)	16. (d)	17. (a,c)	18. (c)		

Topic-3 : Adjoint of a Matrix, Inverse of a Matrix, Some Special Cases of Matrix, Rank of a Matrix

1. (c, d)	2. (c)	3. (4)	4. (a, b, d)	5. (a,b,c)	6. (b,c,d)	7. (a,b,d)	8. (a,c,d)	9. (a,d)	10. (c,d)
11. (c)	12. (d)	13. (b)	14. (a)						

Topic-4 : Solution of System of Linear Equations

1. (a)	2. (b)	3. (a)	4. (b)	5. (d)	6. (d)	7. (1)	8. (1)	9. (1.5)	10. (0)
11. (b)	12. (b, c)	13. (a, d)	14. (b,c,d)	15. (a)	16. (a)	17. (b)	18. (b)	19. (a)	

Hints & Solutions

Topic-1: Minor & Co-factor of an Element of a Determinant, Value of a Determinant

1. **(a)** For the given matrix to be non-singular

$$\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$$

$\Rightarrow 1-(a+c)\,\omega + ac\omega^2 \neq 0 \Rightarrow (1-a\omega)(1-c\omega) \neq 0$

$\Rightarrow a \neq \omega^2$ and $c \neq \omega^2$, where ω is complex cube root of unity.

As a, b and c are complex cube roots of unity.

$\therefore$ a and c can take only one value i.e. ω while b can take 2 values i.e. ω and ω^2.

$\therefore$ Total number of distinct matrices in the set S $= 1 \times 1 \times 2 = 2$

2. **(d)** Given : Three points $P(-\sin(\beta-\alpha), -\cos\beta)$, $Q(\cos(\beta-\alpha), \sin\beta)$ and $R(\cos(\beta-\alpha+\theta), \sin(\beta-\theta))$, where $0 < \alpha, \beta, \theta < \frac{\pi}{4}$

$$\therefore \Delta = \begin{vmatrix} 1 & 1 & 1 \\ -\sin(\beta-\alpha) & \cos(\beta-\alpha) & \cos(\beta-\alpha+\theta) \\ -\cos\beta & \sin\beta & \sin(\beta-\theta) \end{vmatrix}$$

$[C_3 \to C_3 - (C_1 \sin\theta + C_2 \cos\theta)]$

$$\Delta = \begin{vmatrix} 1 & 1 & 1-\sin\theta-\cos\theta \\ -\sin(\beta-\alpha) & \cos(\beta-\alpha) & 0 \\ -\cos\beta & \sin\beta & 0 \end{vmatrix}$$

$= (1-\sin\theta-\cos\theta)[\cos\beta\cos(\beta-\alpha) - \sin\beta\sin(\beta-\alpha)]$

$= [1-(\sin\theta+\cos\theta)]\cos(2\beta-\alpha)$

$\because 0 < \alpha, \beta, \theta < \frac{\pi}{4}$, $\therefore \sin\theta + \cos\theta \neq 1$

Also $2\beta - \alpha < \frac{\pi}{2} \Rightarrow \cos(2\beta-\alpha) \neq 0$

$\therefore \Delta \neq 0 \Rightarrow$ Three given points are non-collinear.

3. **(b)** For every 'determinant, with value 1' ($\in B$) we can find a determinant with value -1 by changing the sign of one entry of '1'.

Hence there are equal number of elements in B and C.

$\therefore$ (b) is the correct option

4. **(16)** $|A| = -(e-d) + c(b-a) = \pm 1$

Case (i) : $c = 0 \Rightarrow e - d = \pm 1$

$\Rightarrow (e, d) = (1, 0), (0, 1) \to 2$ ways

b and a can be each 2 ways

$\Rightarrow$ Total $= 1 \times 2 \times 2 \times 2 = 8$ ways

Case (ii) : $c = 1$

$\Rightarrow d - e + b - a = \pm 1$

c	a	b	d	e
1	1	1	1	0
1	1	1	0	1
1	0	0	1	0
1	0	0	0	1

$\Rightarrow 4 \times 2 = 8$ ways

Total = 16 ways

5. **(5)** $\because$ The trace of A is 3.

Let $A = \begin{bmatrix} x & y \\ z & 3-x \end{bmatrix}$

Now $A^2 = \begin{bmatrix} x & y \\ z & 3-x \end{bmatrix}\begin{bmatrix} x & y \\ z & 3-x \end{bmatrix}$

$= \begin{bmatrix} x^2+yz & xy+3y-xy \\ xz+3z-xz & yz+9+x^2-6x \end{bmatrix}$

$\Rightarrow A^3 = \begin{bmatrix} x^2+yz & 3y \\ 3z & yz+9+x^2-6x \end{bmatrix}\begin{bmatrix} x & y \\ z & 3-x \end{bmatrix}$

$= \begin{bmatrix} x^3+xyz+3yz & x^2y+y^2z+9y-3xy \\ 3xz+yz^2+9z+x^2z-6xz & 6yz+27+3x^2-18x-xyz-9x-x^3+6x^2 \end{bmatrix}$

Given that trace of A^3 is -18

$\therefore x^3 + xyz + 3yz + 6yz + 27 + 3x^2 - 18x - xyz - 9x - x^3 + 6x^2 = -18$

$\Rightarrow$ $9yz + 9x^2 - 27x + 27 = -18 \Rightarrow yz + x^2 - 3x + 3 = -2$

$\Rightarrow$ $3x - x^2 - yz = 5$ ...(i)

Now, $|A| = 3x - x^2 - yz$

$|A| = 5$ From (i)

6. **(4)** Let Det $(P) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

If $a_1 = 1, a_2 = -1, a_3 = 1, b_2c_3 = b_1c_3 = b_1c_2 = 1$ and $b_3c_2 = b_3c_1 = b_2c_1 = -1$

Then maximum value of Det (P) = 6

But it is not possible as

$(b_2c_3)(b_3c_1)(b_1c_2) = -1$ and $(b_1c_3)(b_3c_2)(b_2c_1) = 1$

i.e., $b_1b_2b_3c_1c_2c_3 = 1$ and -1

Similar contradiction occurs when

$a_1 = 1, a_2 = 1, a_3 = 1, b_2c_3 = b_3c_1 = b_1c_2 = 1$ and $b_3c_2 = b_1c_3 = b_2c_1 = -1$

Now, for value to be 5 one of the terms must be zero but that will make 2 terms zero which means answer cannot be 5

Now, $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 4$

Therefore, maximum value is 4.

7. (1) $z=\frac{-1+i\sqrt{3}}{2}\Rightarrow z^3=1$ and $1+z+z^2=0$

$$P^2=\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}$$

$$=\begin{bmatrix}z^{2r}+z^{4s} & z^{2s}\left((-z)^r+z^r\right)\\ z^{2s}\left((-z)^r+z^r\right) & z^{4s}+z^{2r}\end{bmatrix}$$

For $P^2=-I$, we should have

$z^{2r}+z^{4s}=-1$ and $z^{2s}((-z)^r+z^r)=0$

$\Rightarrow z^{2r}+z^{4s}+1=0$ and $(-z)^r+z^r=0$

$\Rightarrow$ r is odd and s = r but not a multiple of 3, which is possible when s = r = 1

$\therefore$ only one pair is there.

8. Given : $\begin{vmatrix}1 & 4 & 20\\ 1 & -2 & 5\\ 1 & 2x & 5x^2\end{vmatrix}=0$

Clearly on expanding the det. we will get a quadratic equation in x.

Therefore, it has 2 roots. We observe that R_3 becomes identical to R_1 if $x=2$. $\therefore$ at $x=2$, $\Delta=0$

Hence, $x=2$ is a root of given equation

Similarly, R_3 becomes identical to R_2 if $x=-1$.

$\therefore$ at $x=-1$, $\Delta=0$

Hence, $x=-1$ is a root of given equation

Therefore, the equation has two roots –1 and 2.

9. As given equation is an identity in λ, it must be true for all values of λ and hence for λ = 0 also.

On putting λ = 0 we get, $t=\begin{vmatrix}0 & -1 & 3\\ 1 & 0 & -4\\ -3 & 4 & 0\end{vmatrix}=0$

10. (a, b) Given : $MN=NM$, $M\neq N^2$ and $M^2=N^4$.

Then $M^2=N^4\Rightarrow(M+N^2)(M-N^2)=0$

$\Rightarrow$ (i) $M+N^2=0$ and $M-N^2\neq 0$

(ii) $|M+N^2|=0$ and $|M-N^2|=0$

In each case $|M+N^2|=0$

$\therefore |M^2+MN^2|=|M|\,|M+N^2|=0$

$\therefore$ (a) is correct and (c) is not correct.

Also we know if $|A|=0$, then there can be many matrices U, such that $AU=0$

$\therefore (M^2+MN^2)U=0$ will be true for many values of U.

$\therefore$ (b) is correct.

Again if $AX=0$ and $|A|=0$, then X can be non-zero.

$\therefore$ (d) is not correct.

11. **A → s; B → p, q; C → r; D → p, q, s**

The given lines are

$L_1: x+3y-5=0$

$L_2: 3x-ky-1=0$

$L_3: 5x+2y-12=0$

(A) Three lines L_1, L_2, L_3 are concurrent, if

$$\begin{vmatrix}1 & 3 & 5\\ 3 & -k & 1\\ 5 & 2 & 12\end{vmatrix}=0\Rightarrow 13k-65=0\Rightarrow k=5$$

$\therefore$ (A) → (s)

(B) For $L_1\parallel L_2\Rightarrow\frac{1}{3}=\frac{-3}{k}\Rightarrow k=-9$

and $L_2\parallel L_3\Rightarrow\frac{3}{5}=\frac{-k}{2}\Rightarrow k=-\frac{6}{5}$

$\therefore$ (B) → (p), (q)

(C) Three lines L_1, L_2, L_3 will form a triangle if no two of them are parallel and no three are concurrent

$\therefore k\neq 5, -9, -6/5$ $\therefore$ (C) → r

(D) Three lines L_1, L_2, L_3 do not form a triangle, if either any two of these are parallel or the three are concurrent i.e.

$k=5, -9, -6/5$ $\therefore$ (D) → (p), (q), (s)

12. (d) Given, $A=\begin{bmatrix}a & b\\ c & a\end{bmatrix}, a, b, c\in\{0,1,2,...,p-1\}$

If A is skew-symmetric matrix, then $a=0$, $b=-c$

$\therefore |A|=-b^2$

Thus, P divides |A|, only when $b=0$. ...(i)

Again, if A is symmetric matrix, then $b=c$ and $|A|=a^2-b^2$

Thus, p divides $|A|$, if either p divides $(a-b)$ or p divides $(a+b)$.

p divides $(a-b)$, only when $a=b$,

i.e. $a=b\in\{0,1,2,...,(p-1)\}$

i.e. p choices ...(ii)

p divides $(a+b)$.

$\Rightarrow p$ choices, including $a=b=0$ included in Eq. (i).

$\therefore$ Total number of choices are $(p+p-1)=2p-1$

13. (c) Trace of $A=2a$, will not divisible by p, iff $a\neq 0$.

$|A|=a^2-bc$, for (a^2-bc) to be divisible by p. There are exactly $(p-1)$ ordered pairs (b, c) for any value of a.

$\therefore$ Required number is $(p-1)^2$.

14. (d) The number of matrices for which p does not divide Tr (A) $=(p-1)p^2$ of these $(p-1)^2$ are such that p divides |A|. The number of materices for which p divides Tr (A) and p does not divides |A| are $(p-1)^2$.

$\therefore$ Required number $=(p-1)p^2-(p-1)^2+(p-1)^2$
$=p^3-p^2$

15. On expanding along R_1, we get

$$\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=a(bc-a^2)-b(b^2-ac)+c(ab-c^2)$$

$=3abc-a^3-b^3-c^3=-(a^3+b^3+c^3-3abc)$

$=-(a+b+c)[a^2+b^2+c^2-ab-bc-ca]$

$=-\frac{1}{2}(a+b+c)[2a^2+2b^2+2c^2-2ab-2bc-2ca]$

$=-\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

As $a, b, c>0$

$\therefore$ $a+b+c>0$

Also $a\neq b\neq c$

$\therefore$ $(a-b)^2+(b-c)^2+(c-a)^2>0$

Hence value of the given determinant is negative.

Topic-2: Properties of Determinants, Area of a Triangle

1. (d) $|Q|=\begin{vmatrix}2^2a_{11} & 2^3a_{12} & 2^4a_{13}\\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23}\\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33}\end{vmatrix}$

$$=2^2.2^3.2^4\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33}\end{vmatrix}$$

$$=2^9.2.2^2\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = 2^{12}\times|P| = 2^{12}\times 2 = 2^{13}$$

2. (c) $A=\begin{bmatrix}\alpha & 2\\ 2 & \alpha\end{bmatrix}$ and $|A^3| = 125 \Rightarrow |A|^3 = 125$

$\therefore$ $|A| = \alpha^2 - 4$, Now $|A|^3 = 125$

$\Rightarrow$ $(\alpha^2-4)^3 = 125 = 5^3 \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$

3. (b) Given that $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$, then $\omega^2 = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

Also $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$

Now $\Delta = \begin{vmatrix}1 & 1 & 1\\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4\end{vmatrix} = \begin{vmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}$

$(\because \omega = -1-\omega^2$ and $\omega^3 = 1)$

$[C_1 \to C_1 + C_2 + C_3]$

$\Delta = \begin{vmatrix}3 & 1 & 1\\ 0 & \omega & \omega^2\\ 0 & \omega^2 & \omega\end{vmatrix}$ $(\because 1+\omega+\omega^2 = 0)$

On expanding along C_1, we get

$\Delta = 3(\omega^2-\omega^4) = 3(\omega^2-\omega) = 3\omega(\omega-1)$

4. (a) $f(x) = \begin{vmatrix}1 & x & x+1\\ 2x & x(x-1) & (x+1)x\\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1)\end{vmatrix}$

$[C_1 \to C_1 + C_2]$

$=\begin{vmatrix}x+1 & x & x+1\\ (x+1)x & x(x-1) & (x+1)x\\ (x+1)x(x-1) & x(x-1)(x-2) & (x+1)x(x-1)\end{vmatrix}$

$= 0$ $[\because C_1$ and C_3 are identical],

which is independent of x, so the function is true for all values of x. $\therefore$ $f(100) = 0$

5. (b) Let $\Delta = \begin{vmatrix}1 & a & a^2\\ \cos(p-d)x & \cos px & \cos(p+d)x\\ \sin(p-d)x & \sin px & \sin(p+d)x\end{vmatrix}$

Applying $C_1 \to C_1 + C_3$

$\Rightarrow \Delta = \begin{vmatrix}1+a^2 & a & a^2\\ 2\cos px\cos dx & \cos px & \cos(p+d)x\\ 2\sin px \cos dx & \sin px & \sin(p+d)x\end{vmatrix}$

$[C_1 \to C_1 - (2\cos dx)C_2]$

$\Delta = \begin{vmatrix}1+a^2-2a\cos dx & a & a^2\\ 0 & \cos px & \cos(p+d)x\\ 0 & \sin px & \sin(p+d)x\end{vmatrix}$

On expanding along C_1, we get

$\Delta = (1+a^2-2a\cos dx)[\sin(p+d)x\cos px - \sin px\cos(p+d)x]$

$\Rightarrow \Delta = (1+a^2-2a\cos dx)[\sin dx]$, which is independent of p.

6. (a) $\begin{vmatrix}1 & 1+i+\omega^2 & \omega^2\\ 1-i & -1 & \omega^2-1\\ -i & -i+\omega-1 & -1\end{vmatrix}$

$[R_1 \to R_1 - R_2 + R_3] = 0$

$=\begin{vmatrix}0 & 0 & 0\\ 1-i & -1 & \omega^2-1\\ -i & -i+\omega+1 & -1\end{vmatrix}$ $(\because 1+\omega+\omega^2 = 0)$

7. (0) Given : $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1+i\sqrt{3}}{2}$

$\therefore 1+\omega+\omega^2 = 0$ and $\omega^3 = 1$

Then $\begin{vmatrix}z+1 & \omega & \omega^2\\ \omega & z+\omega^2 & 1\\ \omega^2 & 1 & z+\omega\end{vmatrix} = 0$ $[C_1 \to C_1 + C_2 + C_3]$

$\Rightarrow \begin{vmatrix}z+1+\omega+\omega^2 & \omega & \omega^2\\ z+1+\omega+\omega^2 & z+\omega^2 & 1\\ z+1+\omega+\omega^2 & 1 & z+\omega\end{vmatrix} = 0$

$\Rightarrow z\begin{vmatrix}1 & \omega & \omega^2\\ 1 & z+\omega^2 & 1\\ 1 & 1 & z+\omega\end{vmatrix} = 0$

$\Rightarrow z\left[1\left(z^2+z\omega+z\omega^2+\omega^3-1\right)-\omega\left(z+\omega-1\right)+\omega^2\left(1-z-\omega^2\right)=0\right]$

$\Rightarrow z\left[z^2\right] = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0$

$\therefore z = 0$ is the only solution.

8. Given that x, y, z are positive numbers, then value of

$D = \begin{vmatrix}1 & \log_x y & \log_x z\\ \log_y x & 1 & \log_y z\\ \log_z x & \log_z y & 1\end{vmatrix}$

$= \begin{vmatrix}1 & \frac{\log y}{\log x} & \frac{\log z}{\log x}\\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y}\\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1\end{vmatrix}$ $\left(\because \log_b a = \frac{\log a}{\log b}\right)$

Taking $\frac{1}{\log x}, \frac{1}{\log y}$ and $\frac{1}{\log z}$ common from R_1, R_2 and R_3 respectively

$$D=\frac{1}{\log x\log y\log z}\begin{vmatrix}\log x & \log y & \log z\\ \log x & \log y & \log z\\ \log x & \log y & \log z\end{vmatrix}=0$$

9. $\begin{vmatrix}1 & a & a^2-bc\\ 1 & b & b^2-ca\\ 1 & c & c^2-ab\end{vmatrix}$ $[R_1\to R_1-R_2;\ R_2\to R_2-R_3]$

$$\begin{vmatrix}0 & a-b & (a-b)(a+b+c)\\ 0 & b-c & (b-c)(a+b+c)\\ 1 & c & c^2-ab\end{vmatrix}$$

$$=(a-b(b-c)\begin{vmatrix}0 & 1 & a+b+c\\ 0 & 1 & a+b+c\\ 1 & c & c^2-ab\end{vmatrix}=0$$

10. $\begin{vmatrix}x & 3 & 7\\ 2 & x & 2\\ 7 & 6 & x\end{vmatrix}=0$, $[R_1\to R_1+R_2+R_3]$

$$\begin{vmatrix}x+9 & x+9 & x+9\\ 2 & x & 2\\ 7 & 6 & x\end{vmatrix}=0\Rightarrow(x+9)\begin{vmatrix}1 & 1 & 1\\ 2 & x & 2\\ 7 & 6 & x\end{vmatrix}=0$$

$[C_2\to C_2-C_1, C_3\to C_3-C_1]$

$$\Rightarrow(x+9)\begin{vmatrix}1 & 0 & 0\\ 2 & x-2 & 0\\ 7 & -1 & x-7\end{vmatrix}=0$$

Expanding along R_1

$\Rightarrow$ $(x+9)(x-2)(x-7)=0\Rightarrow x=-9, 2, 7$

$\therefore$ Other roots are 2 and 7.

11. **(False)** $\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & 1\\ a_2 & b_2 & 1\\ a_3 & b_3 & 1\end{vmatrix}$

$$\Rightarrow\ \frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}a_1 & b_1 & 1\\ a_2 & b_2 & 1\\ a_3 & b_3 & 1\end{vmatrix}$$

$\Rightarrow$ area (Δ_1) = area (Δ_2), where Δ_1 is the area of triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) and Δ_2 is the area of triangle with vertices (a_1, b_1), (a_2, b_2) and (a_3, b_3). But two Δ's of same area may not be congruent.

Hence, the given statement is false.

12. **(False)** $\begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}=\frac{1}{abc}\begin{vmatrix}a & a^2 & abc\\ b & b^2 & abc\\ c & c^2 & abc\end{vmatrix}$

$$=\begin{vmatrix}a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix}=(-1)^2\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$[C_1\Leftrightarrow C_3$ and then $C_2\Leftrightarrow C_3]$

$=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$. Hence statement is false.

13. **(a, c)**

$$\because f(\theta)=\frac{1}{2}\begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}+$$

$$\begin{vmatrix}\sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right)\\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi\end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix}2 & \sin\theta & 1\\ 0 & 1 & \sin\theta\\ 0 & -\sin\theta & 1\end{vmatrix}+$$

$$\begin{vmatrix}0 & -\sin\left(\theta-\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & 0 & \log_e\left(\frac{4}{\pi}\right)\\ -\tan\left(\theta-\frac{\pi}{4}\right) & -\log_e\left(\frac{4}{\pi}\right) & 0\end{vmatrix}$$

Here $\cos\left(\theta+\frac{\pi}{4}\right)=-\sin\left(\theta-\frac{\pi}{4}\right)$

and $\tan\left(\theta-\frac{\pi}{4}\right)=-\cot\left(\theta+\frac{\pi}{4}\right)$

and $\log_e\left(\frac{4}{\pi}\right)=-\log_e\left(\frac{\pi}{4}\right)$

Also $\sin\pi=-\cos\frac{\pi}{2}=\tan\pi=0$

$\therefore$ $f(\theta)=1+\sin^2\theta$

$\therefore$ $g(\theta)=|\sin\theta|+|\cos\theta|$

$\therefore$ maximum and minimum values are $\sqrt{2}$ and 1 respectively

$g(\theta)\in\left[1,\sqrt{2}\right]$

$\therefore$ $P(x)=a\left(x-\sqrt{2}\right)(x-1)$, where $a\in R-\{0\}$,

But $P(2)=2-\sqrt{2}$ then $a=1$

$\therefore$ $P(x)=\left(x-\sqrt{2}\right)(x-1)$

$\therefore \quad P\left(\frac{3+\sqrt{2}}{4}\right)=\left(\frac{3-3\sqrt{2}}{4}\right)\cdot\left(\frac{\sqrt{2}-1}{4}\right)<0$

$\therefore \quad P\left(\frac{1+3\sqrt{2}}{4}\right)=\left(\frac{1-\sqrt{2}}{4}\right)\cdot\left(\frac{3\sqrt{2}-3}{4}\right)<0$

$P\left(\frac{5\sqrt{2}-1}{4}\right)=\left(\frac{\sqrt{2}-1}{4}\right)\cdot\left(\frac{5\sqrt{2}-5}{4}\right)>0$

$P\left(\frac{5-\sqrt{2}}{4}\right)=\left(\frac{5-5\sqrt{2}}{4}\right)\left(\frac{1-\sqrt{2}}{4}\right)>0$

$\therefore$ (a) and (c) and correct.

14. **(b, d)** In options (a) and (c) $|A^2| = 1$ and in option (b) and (d) $|A^2| = -1$
Since $|A^2| = |A|^2$ and $|A|^2 \neq -1$
$\Rightarrow$ Matrices given in options (b) and (d) cannot be the squares of any 3×3 matrix with real entries.

15. **(b, c)** $\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2\\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2\end{vmatrix}$

$[R_2 \to R_2 - R_1, R_3 \to R_3 - R_2]$

$$\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ 2\alpha+3 & 4\alpha+3 & 6\alpha+3\\ 2\alpha+5 & 4\alpha+5 & 6\alpha+5\end{vmatrix}=-648\alpha$$

$[R_3 \to R_3 - R_2]$

$$2\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ 2\alpha+3 & 4\alpha+3 & 6\alpha+3\\ 1 & 1 & 1\end{vmatrix}=-648\alpha$$

$[C_2 \to C_2 - C_1, C_3 \to C_3 - C_2]$

$$\begin{vmatrix}(1+\alpha)^2 & \alpha(3\alpha+2) & \alpha(5\alpha+2)\\ 2\alpha+3 & 2\alpha & 2\alpha\\ 1 & 0 & 0\end{vmatrix}=-324\alpha$$

$\therefore \quad 2\alpha^2(-2\alpha) = -324\alpha \Rightarrow \alpha^3 - 81\alpha = 0 \Rightarrow \alpha = 0, 9, -9$

16. **(d)** $\begin{vmatrix}6i & -3i & 1\\ 4 & 3i & -1\\ 20 & 3 & i\end{vmatrix}=x+iy \Rightarrow -3i\begin{vmatrix}6i & 1 & 1\\ 4 & -1 & -1\\ 20 & i & i\end{vmatrix}=x+iy$

$\Rightarrow \quad 0 = x+iy \qquad$ [$\because$ C_2 and C_3 are identical] $\therefore x = 0, y = 0$

17. **(a, c)** $\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta\end{vmatrix}=0$

Applying $C_1 \to C_1 + C_2$

$$\begin{vmatrix}2 & \cos^2\theta & 4\sin 4\theta\\ 2 & 1+\cos^2\theta & 4\sin 4\theta\\ 1 & \cos^2\theta & 1+4\sin 4\theta\end{vmatrix}=0$$

Applying $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$

$$\begin{vmatrix}0 & -1 & 0\\ 1 & 1 & -1\\ 1 & \cos^2\theta & 1+4\sin 4\theta\end{vmatrix}=0$$

$\Rightarrow \quad [1 + 4\sin 4\theta + 1] = 0$

$\Rightarrow \quad 2(1 + 2\sin 4\theta) = 0 \Rightarrow \sin 4\theta = -\frac{1}{2}$

$\Rightarrow \quad 4\theta = \pi + \pi/6$ or $2\pi - \pi/6$

$\Rightarrow \quad 4\theta = 7\pi/6$ or $11\pi/6 \Rightarrow \theta = 7\pi/24$ or $11\pi/24$

18. **(c)** α, β are roots of $x^2 + x - 1 = 0$
$\therefore \alpha + \beta = -1 \Rightarrow 1 + \alpha + \beta = 0$

$$M=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

(P) $M=\begin{bmatrix}1 & \alpha & \beta\\ \alpha & \beta & 1\\ \beta & 1 & \alpha\end{bmatrix} \Rightarrow 3!\times 2 = 12$

For one arrangement of row 1 we can arrange other two rows exactly in two ways and row 1 can be arranged in 3! ways
$\therefore 3! \times 2 = 12$ ways

(Q) $M=\begin{bmatrix}1 & \alpha & \beta\\ \alpha & \beta & 1\\ \beta & 1 & \alpha\end{bmatrix}$

1, α, β can be arranged in 3! ways and corresponding entries can be arranged in 1 way.

(R) $\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ 0\\ -c\end{bmatrix}$

$ay + bz = a$
$-ax + cz = 0$
$-bx - cy = -c$
It is observed that $D = D_x = D_y = D_z = 0$
$\therefore$ infinite solution

(S) $\begin{bmatrix}1 & \alpha & \beta\\ \beta & \alpha & 1\\ \alpha & 1 & \beta\end{bmatrix}$

$\Rightarrow \alpha\beta = 1 - \alpha\beta^2 + \alpha^2 + \beta^2 - \alpha^2\beta = 0$ (since $\alpha\beta = \alpha + \beta = -1$)

19. Given : $MM^T = I$, where M is a square matrix of order 3 and det $M = 1$.
Now det $(M - I)$ = det $(M - MM^T)$ $\quad$ [$\because MM^T = I$]
$\quad$ = det $[M(I - M^T)]$
$\quad$ = (det M) (det $(I - M^T)$) $\quad$ [$\because |AB| = |A||B|$]
$\quad$ = – (det M) (det $(M^T - I)$)
$\quad$ = – [det $(M^T - I)$] $\quad$ [$\because$ det (M) = 1]
$\Rightarrow$ det $(M - I)$ = – det $(M - I)$
[$\because$ det $(M^T - I)$ = det $[(M - I)^T]$ = det $(M - I)$]
$\Rightarrow$ 2 det $(M - I) = 0 \Rightarrow$ det $(M - I) = 0$

20. Given : $A^TA = I$

$\Rightarrow |A^TA| = |A^T||A| = |A||A| = 1 \quad [\because |I| = I]$

$\Rightarrow |A|^2 = 1$

Now $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$\Rightarrow |A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

$\because |A|^2 = 1, \therefore (a^3 + b^3 + c^3 - 3abc)^2 = 1$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 1$ or -1

$\because AM \geq GM, \therefore \dfrac{a^3+b^3+c^3}{3} \geq \sqrt[3]{a^3b^3c^3}$

$\Rightarrow a^3 + b^3 + c^3 - 3abc \geq 0$

$\therefore a^3 + b^3 + c^3 - 3abc = 1$

$\Rightarrow a^3 + b^3 + c^3 = 1 + 3 \times 1 = 4 \quad [\because abc = 1]$

21. LHS $= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$

$[R_2 \to R_2 + R_3]$

$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cos\frac{2\pi}{3} & 2\cos\theta\cos\frac{2\pi}{3} & 2\sin 2\theta\cos\frac{4\pi}{3} \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$

$= -\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$

[Since R_1 and R_2 are identical]

22. Let us denote the given determinant by Δ. On taking common $\dfrac{1}{a(a+d)(a+2d)}$ from R_1,

$\dfrac{1}{(a+d)(a+2d)(a+3d)}$ from R_2

and $\dfrac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3, we get

$\Delta = \dfrac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta_1$, where

$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$

$[R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1]$

$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$

$[R_3 \to R_3 - R_2]$

$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$

On expanding along R_3, we get

$\Delta_1 = (2d)^2 (d) (a + 2d - a) = 4d^4$

$\therefore \Delta = \dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

23. L.H.S.

$= \begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos(A-Q) & \cos(A-R) \\ \cos B\cos P + \sin B\sin P & \cos(B-Q) & \cos(B-R) \\ \cos C\cos P + \sin C\sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$= \cos P\begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P\begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$= \cos P\begin{vmatrix} \cos A & \sin A\sin Q & \sin A\sin R \\ \cos B & \sin B\sin Q & \sin B\sin R \\ \cos C & \sin C\sin Q & \sin C\sin R \end{vmatrix} + \sin P\begin{vmatrix} \sin A & \cos A\cos Q & \cos A\cos R \\ \sin B & \cos B\cos Q & \cos B\cos R \\ \sin C & \cos C\cos Q & \cos C\cos R \end{vmatrix}$

[$C_2 \to C_2 - C_1 (\cos Q)$; $C_3 \to C_3 - C_1 (\cos R)$ on first determinant and $C_2 \to C_2 - (\sin Q)C_1$; $C_3 \to C_3 - (\sin R)C_1$ on second determinant.]

$= \cos P\sin Q\sin R\begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} + \sin P\cos Q\cos R\begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix}$

$= 0 + 0$ [Both determinants become zero as $C_2 \equiv C_3$]

= R.H.S.

24. $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$

$\Rightarrow D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$

$[R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2]$

$$\Rightarrow D = (n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

$[R_3 \to R_3 - R_2]$

$$\Rightarrow D = (n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\Rightarrow D = (n!)^3 (n+1)^2 (n+2)\, 2 \Rightarrow \frac{D}{(n!)^3} = 2\ (n+1)^2 (n+2)$$

$$\Rightarrow \frac{D}{(n!)^3} - 4 = 2\,(n^3 + 4n^2 + 5n + 2) - 4$$

$$\Rightarrow \frac{D}{(n!)^3} = 2\,(n^3 + 4n^2 + 5n) = 2n\,(n^2 + 4n + 5)$$

$\therefore \left(\frac{D}{(n!)^3} - 4\right)$ is divisible by n.

25. $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$ $[R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3]$

$$\Rightarrow \begin{vmatrix} p-a & -(q-b) & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

On taking $(p-q)$, $(q-b)$ and $(r-c)$ common from C_1, C_2 and C_3 respectively, we get

$$(p-a)(q-b)(r-c) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a}{p-a} & \frac{b}{q-b} & \frac{r}{r-c} \end{vmatrix} = 0$$

On expanding along R_1, we get

$$(p-a)(q-b)(r-c)\left[1\left(\frac{r}{r-c} + \frac{b}{q-b}\right) + \frac{a}{p-a}\right] = 0$$

$\because p \neq a, q \neq b, r \neq c$

$$\therefore \quad \frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0$$

$$\Rightarrow \frac{r}{r-c} + \frac{q-(q-b)}{q-b} + \frac{p-(p-a)}{p-a} = 0$$

$$\Rightarrow \frac{r}{r-c} + \frac{q}{q-b} - 1 + \frac{p}{p-a} - 1 = 0$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

26. Given : A, B, C are integers between 0 and 9 and the three digit numbers $A28$, $3B9$ and $62C$ are divisible by a fixed integer k.

Now, $D = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

$[R_2 \to R_2 + 10\,R_3 + 100\,R_1]$

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix} = \begin{vmatrix} A & 3 & 6 \\ kn_1 & kn_2 & kn_3 \\ 2 & B & 2 \end{vmatrix}$$

[$\because$ $A28$, $3B9$ and $62\,C$ are divisible by k,
$\therefore$ $A28 = kn_1$; $3B9 = kn_2$; $62C = kn_3$, n_1, n_2, n_3 are integers]

$$= k\begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix} = k \times \text{some integral value.}$$

$\therefore$ D is divisible by k.

27. $\Delta a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$\therefore \quad \sum_{a=1}^{n} \Delta a = \begin{vmatrix} (1-1) & n & 6 \\ (1-1)^2 & 2n^2 & 4n-2 \\ (1-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$+ \begin{vmatrix} (2-1) & n & 6 \\ (2-1)^2 & 2n^2 & 4n-2 \\ (2-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} + \ldots + \begin{vmatrix} (n-1) & n & 6 \\ (n-1)^2 & 2n^2 & 4n-2 \\ (n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} 1+2+3+\ldots+(n-1) & n & 6 \\ 1^2+2^2+3^2+\ldots+(n-1)^2 & 2n^2 & 4n-2 \\ 1^3+2^3+3^3+\ldots+(n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \left(\frac{n(n-1)}{2}\right)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n^2(n-1)}{12} \begin{vmatrix} 6 & 1 & 6 \\ 2(2n-1) & 2n & 2(2n-1) \\ 3n(n-1) & 3n^2 & 3n(n-1) \end{vmatrix}$$

[Taking $\frac{n(n-1)}{12}$ common from C_1 and n from C_2]

$= 0$ [$\because$ C_1 and C_3 are identical]

$\because \sum_{a=1}^{n} \Delta a = 0 \Rightarrow \sum_{a=1}^{n} \Delta a = c$ (a constant) where $c = 0$

28. LHS = $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$

$C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_2$ and using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$, we get

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

Operating $C_3 \to C_3 + C_2$ and using the same result, we get

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} = \text{RHS}$$

29. $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$

$$\text{L.H.S.} = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

$[R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1]$

$$= \begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x+1 & x-2 \\ 0 & x-2 & x+1 \\ 0 & x-2 & x+1 \end{vmatrix} + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$[R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1]$

$$= \begin{vmatrix} x & x+1 & x-2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} = \begin{vmatrix} x & x & x \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

$= xA + B =$ R.H. S

Topic-3: Adjoint of a Matrix, Inverse of a Matrix, Some Special Cases of Matrix, Rank of a Matrix

1. **(c, d)** Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where a, b, c are integers.

M is invertible, if $\begin{vmatrix} a & b \\ b & c \end{vmatrix} \neq 0 \Rightarrow ac \neq b^2$

$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \Rightarrow a = b = c \Rightarrow ac = b^2.$

Hence (a) is not correct.

If $\begin{bmatrix} b & c \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix} \Rightarrow b = a = c \Rightarrow ac = b^2$

$\therefore$ (b) is not correct.

If M = $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$, then $|M| = ac \neq 0$

$\therefore$ M is invertible.

(c) is correct

Since, $ac \neq (\text{integer})^2 \Rightarrow ac \neq b^2$

$\therefore$ (d) is correct.

2. **(c)** Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$

$\therefore$ Characteristic eqn of above matrix A is given by

$$|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{bmatrix} = 0$$

$\Rightarrow (1-\lambda)(4 - 5\lambda + \lambda^2 + 2) = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Also by Cayley Hamilton theorem, every square matrix satisfies its characteristic equation.

$\therefore \quad A^3 - 6A^2 + 11A - 6I = 0$

On multiplying by A^{-1}, we get

$A^2 - 6A + 11I - 6A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$

On comparing it with given relation,

$$A^{-1} = \frac{1}{6}(A^2 - cA + dI)$$

We get $c = -6$ and $d = 11$

3. **(4)** $|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 2\sqrt{k} & 1+2k & -2k \\ -2\sqrt{k} & 1+2k & -1 \end{vmatrix} \quad [C_2 \to C_2 - C_3]$$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 4\sqrt{k} & 0 & 1-2k \\ -2\sqrt{k} & 1+2k & -1 \end{vmatrix} \quad [R_2 \to R_2 - R_3]$$

$= (1+2k)(8k-4k+4k^2+1) = (2k+1)^3$

Since B is skew symmetric of odd order,

$\therefore |B| = 0$

Hence, $|\text{Adj A}| + |\text{Adj B}| = |A|^2 + |B|^2 = 10^6$

$\Rightarrow (2k+1)^6 = 10^6 \Rightarrow k = 4.5, \quad \therefore [k] = 4$

4. **(a, b, d)**
(a) We have

$$PEP = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&4\\8&13&18\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\8&13&18\\2&3&4\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&3&2\\8&18&13\\2&4&3\end{pmatrix} = F$$

$$P^2 = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

(b) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$
$|E| = 0$ and $|F| = 0$ and $|Q| \neq 0$

$$|EQ| = |E||Q| = 0, \quad |PFQ^{-1}| = \frac{|P||F|}{|Q|} = 0$$

$$\left[\because |Q^{-1}| = \frac{1}{|Q|}\right]$$

$T = EQ + PFQ^{-1}$
$TQ = EQ^2 + PF = EQ^2 + P^2EP$
$= EQ^2 + EP = E(Q^2 + P)$ [from (a)]
$|TQ| = |E(Q^2+P)| \Rightarrow |T||Q| = |E||Q^2+P| = 0$
$\Rightarrow |T| = 0$ (as $|Q| \neq 0$)

(c) $|(EF)^3| > |EF|^2$
Here $0 > 0$ (false)

(d) as $P^2 = I \Rightarrow P^{-1} = P$
So, $Tr(P^{-1}EP + F) = Tr(PEP + F) = Tr(2F)$
Also $Tr(E + P^{-1}FP) = Tr(E + PFP) = Tr(2E)$
Given that
$Tr(E) = Tr(F)$
$\therefore Tr(2E) = Tr(2F)$
$Tr(2PEP) = 2Tr(PEP) = 2Tr(EPP) = 2Tr(E)$

5. **(a, b, c)** Since $(I - EF)$ is invertible therefore,
$|I - EF| \neq 0$; $G = (I - EF)^{-1} \Rightarrow G^{-1} = I - EF$
We have $G.G^{-1} = I = G^{-1}G$
$\Rightarrow G(I - EF) = I = (I - EF)G$
$\Rightarrow G - GEF = I = G - EFG \ldots$ (i)
$\Rightarrow$ GEF = EFG (Option (c) is correct)
$(I - FE)(I + FGE) = I + FGE - FE - FEFGE$
$= I + FGE - FE - F(G - I)E$ [from (i)]
$= I + FGE - FE - FGE + FE$
$= I$ [Option (b) is correct]
$(I - FE)(I + FGE) = I$ . . . (ii)
(So, option 'd' is incorrect)
Now $FE(I + FGE)$
$= FE + FEFGE$
$= FE + F(G - I)E$ [form (i)]
$= FE + FGE - FE = FGE$
$\Rightarrow |FE||I + FGE| = |FGE|$

$$\Rightarrow |FE| \times \frac{1}{|I - FE|} = |FGE| \text{ (from (ii))}$$

$\Rightarrow |FE| = |I - FE||FGE|$
(Option (a) is correct)

6. **(b,c,d)** Since, matrix M is invertible matrix
Then,
det (M) $\neq 0$
$M^{-1} = \text{adj (adj M)}$
$M^{-1} = \det(M).M$ $[\because \text{adj(adj M)} = (\det M)^{3-2}.\ M]$
$M^{-1}M = \det(M).M^2$
$I = \det(M).M^2$ ...(i)
$\det(I) = (\det(M))^5$
$[\because \det(\det M) = (\det M)^3$ and $\det(M^2) = (\det M)^2$
$I = \det(M)$ ...(ii)
From (i), $I = M^2$
$(\text{adj M})^2 = \text{adj}(M^2) = \text{adj I} = I.$

7. **(a,b, d)** We observe that $P_1' = P_1, P_2' = P_2, P_3' = P_3,$

$$P_4' = P_5, P_5' = P_4, P_6' = P_6$$

Also $P_kP_k' = I$ for $k = 1$ to 6.

$$\text{Now } X = \sum_{k=1}^{6} P_kAP_k', \text{ where } A = \begin{bmatrix}2&1&3\\1&0&2\\3&2&1\end{bmatrix} \text{ and } A' = A$$

$$X' = \left(\sum_{k=1}^{6} P_kAP_k'\right)' = \sum_{k=1}^{6}\left(P_kAP_k'\right)'$$

$$= \sum_{k=1}^{6}(P'_k)'A'P_x' = \sum_{k=1}^{6} P_k AP_k' = X$$

$\therefore$ X is a symmetric matrix.
$\therefore$ (a) is correct.
Sum of diagonals entries of X = Trace X

$$= \text{Trace}\sum_{k=1}^{6}(P_kAP_k') = \sum_{k=1}^{6}\text{Trace}(P_kAP_k')$$

$$= \sum_{k=1}^{6}\text{Trace}(AP_kP_k') \text{ [using Trace AB = Trace BA]}$$

$$= \sum_{k=1}^{6}\text{Trace}(AI) = \sum_{k=1}^{6}\text{TraceA} = 6 \times \text{TraceA}$$

$= 6 \times (2 + 0 + 1) = 18$
$\therefore$ (b) is correct

$$X\begin{bmatrix}1\\1\\1\end{bmatrix} = \sum_{k=1}^{6}P_kA\left(P_k'\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \sum_{k=1}^{6}P_kA\begin{bmatrix}1\\1\\1\end{bmatrix} = \sum_{k=1}^{6}P_k\begin{bmatrix}6\\3\\6\end{bmatrix}$$

$$= \begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}\begin{bmatrix}6\\3\\6\end{bmatrix} = \begin{bmatrix}30\\30\\30\end{bmatrix} = 30\begin{bmatrix}1\\1\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\therefore \alpha = 30$
$\therefore$ (d) is correct.

$$\text{Also } X\begin{bmatrix}1\\1\\1\end{bmatrix} = 30\begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow (X - 30\,I)\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$$

$\Rightarrow X - 30\,I = 0 \Rightarrow |X - 30\,I| = 0$
$\Rightarrow X - 30\,I$ is not invertible
$\therefore$ (c) is incorrect.

8. **(a, c, d)** $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$; Adj $M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

Cofactor of a_{11} in $M = 2 - 3b = -1 = a_{11}$ in Adj $M \Rightarrow b = 1$

Cofactor of a_{31} in $M = 3 - 2a = -1 = a_{13}$ in Adj $M \Rightarrow a = 2$

$\therefore\ a + b = 2 + 1 = 3$

$\therefore$ (a) is correct.

$$|M| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -(1-9) + 2(1-6) = 8 - 10 = -2$$

$|\text{adj } M^2| = |M^2|^2 = |M|^4 = (-2)^4 = 16$

$\therefore$ (b) is incorrect.

Also $(\text{adj } M)^{-1} = \text{adj } M^{-1}$

$\therefore\ (\text{adj } M)^{-1} + \text{adj } M^{-1} = 2\text{adj }(M^{-1})$

$$= \frac{2|M^{-1}|}{|M^{-1}|}\text{adj}(M^{-1}) = 2|M^{-1}| \cdot (M^{-1})^{-1} = 2 \times \frac{1}{-2} \times M = -M$$

$\therefore$ (c) is correct.

$$\text{Now, } M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = M^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore\ \alpha - \beta + \gamma = 1 - (-1) + 1 = 3$

$\therefore$ (d) is correct.

9. **(a,d)** $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}; Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$

$R = PQP^{-1}$

$\Rightarrow \det R = \det(PQP^{-1}) = |P|\,|Q|\,|P^{-1}| = |Q| = 4(12 - x^2)$

$$\text{Also, } \det\begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8 = 4(12 - x^2) \quad \left[\because |P^{-1}| = \frac{1}{|P|}\right]$$

$$\therefore \det R = \det\begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8\ \forall x \in R$$

$\therefore$ (a) is correct

For $x = 1$, $\det R = 4(12 - 1) \neq 0$

$$\therefore R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ will have only trivial solution}$$

i.e., $\alpha = 0$, $\beta = 0$ and $\gamma = 0$

$\therefore$ (b) is incorrect

For $PQ = QP$, a_{11} in $PQ = a_{11}$ in QP

$\Rightarrow 2 + x = 2 \Rightarrow x = 0$

Now a_{12} in $PQ = 2x + 4$ and a_{12} in $QP = 2 + 2x$

$\Rightarrow 2x + 4 = 2 + 2x \Rightarrow 4 = 2$, which is not possible

$\therefore$ $PQ = QP$, is not possible for any real x.

$\therefore$ (c) is incorrect.

$$\text{For } x = 0,\ R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\text{Now } R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 6a \\ 6b \end{bmatrix}$$

$\Rightarrow 2 + a + \frac{2}{3}b = 6 \Rightarrow 3a + 2b = 12$ .... (i)

and $4a + \frac{4}{3}b = 6a \Rightarrow 3a - 2b = 0$ .... (ii)

On solving (i) and (ii), we get $a = 2$, $b = 3 \Rightarrow a + b = 5$

$\therefore$ (d) is correct

10. **(c, d)** (a) $(N'MN)' = (MN)'N = N'M'N = N'MN$ or $-N'MN$, according as M is symmetric or skew symmetric matrix. Hence (a) is coorect

(b) $(MN - NM)' = (MN)' - (NM)' = N'M' - M'N'$
$= NM - MN = -(MN - NM)$

Hence $(MN - NM)$ is skew symmetric matrix. Therefore (b) is correct.

(c) $(MN)' = N'M' = NM \neq MN$

Hence (c) is incorrect.

(d) $(\text{adj } M)(\text{adj } N) = \text{adj }(MN)$ is incorrect.

11. **(c)** [Since a skew symmetric matrix of order 3 cannot be non singular, therefore the data given in the question is inconsistent.]

Now $M^2N^2(M^TN)^{-1}(MN^{-1})^T$

$= M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^TM^T$

$= M^2N(M^T)^{-1}(N^{-1})^TM^T = -M^2NM^{-1}N^{-1}M$

$[\because M^T = -M, N^T = -N \text{ and } (N^{-1})^T = (N^T)^{-1}]$

$= -M(NM)(NM)^{-1}M$ $\quad[\because MN = NM]$

$= -MM = -M^2$

12. **(d)** Let $U_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\text{Now } AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ 2a + b \\ 3a + 2b + c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a = 1, b = -2, c = 1$$

$$\therefore U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ Similarly, } U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}, U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \Rightarrow |U| = 3$$

13. **(b)** $U^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$

$\Rightarrow$ Sum of elements of $U^{-1} = \frac{1}{3}(0) = 0$

14. **(a)** $[3\ 2\ 0]\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = [3\ 2\ 0]\begin{bmatrix} 7 \\ -8 \\ -5 \end{bmatrix} = 5$

Topic-4: Solution of System of Linear Equations

1. **(a)** Since, $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is linear equation in three variables and that could have only unique, no solution or infinitely many solution.

$\therefore$ It is not possible to have two solutions.
Hence, number of matrices A is zero.

2. **(b)** Given system : $2x - y + 2z = 2$, $x - 2y + z = -4$ and $x + y + \lambda z = 4$

Since the system has no solution, $\Delta = 0$ and any one amongst $\Delta_x, \Delta_y, \Delta_z$ is non-zero.

Now $\Delta = 0 \Rightarrow \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1$

Also, $\Delta_z = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -4 \\ 1 & 1 & 4 \end{vmatrix} = 6 \neq 0$

3. **(a)** Given system : $x + ay = 0$, $az + y = 0$, $ax + z = 0$
It is system of homogeneous equations, therefore it will have infinite many solutions if determinant of coefficient matrix is zero.

i.e., $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1(1-0) - a(0 - a^2) = 0$

$\Rightarrow 1 + a^3 = 0 \Rightarrow a^3 = -1 \Rightarrow a = -1$

4. **(b)** For infinitely many solutions,

$\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1} \Rightarrow k = 1$

5. **(d)** For the given homogeneous system to have non zero solution, determinant of coefficient matrix should be zero

i.e., $\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow 1(1+1) + k(-k+1) - 1(k+1) = 0$
$\Rightarrow 2 - k^2 + k - k - 1 = 0 \Rightarrow k = \pm 1$

6. **(d)** Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y, \frac{z^2}{c^2} = Z$. Then the given system of equations becomes
$X + Y - Z = 1$, $X - Y + Z = 1$, $-X + Y + Z = 1$
For this new system, we have

$D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4 \neq 0$

$\therefore$ New system of equations has unique solution.

$D_1 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4$, $D_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4$

and $D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4$

Now, $X = \frac{D_1}{D} = \frac{-4}{-4} = 1$, $Y = \frac{D_2}{D} = \frac{-4}{-4} = 1$ and

$Z = \frac{D_3}{D} = \frac{-4}{-4} = 1 \Rightarrow x = \pm a,\ y = \pm b$ and $z = \pm c$

7. **(1)** For infinite many solutions

$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0 \Rightarrow (1-\alpha^2)^2 = 0 \Rightarrow \alpha = \pm 1$

For $\alpha = 1$, the system will have no solution and for $\alpha = -1$, all three equations giving infinite many dependent solutions.

$\therefore\quad 1 + \alpha + \alpha^2 = 1 - 1 + 1 = 1$

Solution 8 and 9 :

$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

Given that system of equations is consistent.

$\therefore\ \Delta_x = \begin{vmatrix} \alpha & 2 & 3 \\ \beta & 5 & 6 \\ \gamma-1 & 8 & 9 \end{vmatrix} = 0 \Rightarrow \alpha - 2\beta + \gamma = 1$

$|M| = \begin{vmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \alpha - 2\beta + \gamma = 1$

The plane containing all those (α, β, γ) :
$P : x - 2y + z = 1$

Perpendicular distance from $(0, 1, 0) = \left|\frac{3}{\sqrt{6}}\right| = x$

$\Rightarrow D = x^2 = \frac{9}{6} = 1.5$

8. **(1.00)** **9.** **(1.50)**

10. The given homogeneous system of equations will have non zero solution if $D = 0$

$\Rightarrow \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(\lambda^2 + 1) - 1(-\lambda + 1) + 1(1 + \lambda) = 0 \Rightarrow \lambda^3 + 3\lambda = 0$
$\Rightarrow \lambda(\lambda^2 + 3) = 0$, but $\lambda^2 + 3 \neq 0$ for real $\lambda \Rightarrow \lambda = 0$

11. **(b)** We have system of dinear equations

$x+y+z=1$ ...(i)

$10x+100y+1000z=0$

$x+10y+100z=0$ ...(ii)

$qrx+pry+pqz=0$ ...(iii)

$\Rightarrow \quad \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=0 \quad (\because p,q,r\neq 0)$

Let $p=\frac{1}{a+9d}, q=\frac{1}{a+99d}, r=\frac{1}{a+999d}$

Now, equation (iii) is

$(a+9d)x+(a+99d)y+(a+999d)z=0$

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 10 & 100\\ a+9d & a+99d & a+999d\end{vmatrix}=0$$

$$\Delta_x=\begin{vmatrix}1 & 1 & 1\\ 0 & 10 & 100\\ 0 & a+99d & a+999d\end{vmatrix}=900(d-a)$$

$$\Delta_y=\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 100\\ a+9d & 0 & a+999d\end{vmatrix}=990(a-d)$$

$$\Delta_z=\begin{vmatrix}1 & 1 & 1\\ 1 & 10 & 0\\ a+9d & a+99d & 0\end{vmatrix}=90(d-a)$$

Let option I: If $\frac{q}{r}=10 \Rightarrow a=d$

$\Delta=\Delta_x=\Delta_y=\Delta_z=0$

Since eq. (i) and eq. (ii) represents non-parallel planes and eq. (ii) and eq. (iii) represents same plane

$\Rightarrow$ Infinitely many solutions

So, option I $\to P, Q, R, T$

Option II: $\frac{p}{r}\neq 100 \Rightarrow a\neq d$

$\Delta=0, \Delta_x, \Delta_y, \Delta_z\neq 0$

No solution

So, option II $\to$ S

Option III: $\frac{p}{q}\neq 10 \Rightarrow a\neq d$

No solution

So, option III $\to$ S

Option IV: If $\frac{p}{q}=10 \Rightarrow a=d$

Infinitely many solution

Hence, IV $\to$ P, Q, R, T

12. **(b, c)** Given that $a_{ij}=-1$ if j + 1 is divisible by i, otherwise $a_{ij}=0$

$$M=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

(a) $|M|=-1+1=0 \Rightarrow$ M is singular so non-invertible.

So, option (a) is incorrect.

(b) $M\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}-a_1\\ -a_2\\ -a_3\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}-a_1\\ -a_2\\ -a_3\end{bmatrix}$

$$\left.\begin{aligned}a_1+a_2+a_3&=-a_1\\ a_1+a_3&=-a_2\\ a_2&=-a_3\end{aligned}\right\}\Rightarrow a_1=0 \text{ and } a_2+a_3=0 \text{ infinite}$$

solutions. So, option (b) is correct.

(c) $MX=0\Rightarrow\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$x+y+z=0$

$x+z=0$

$y=0$

$\therefore$ Infinite solution. So, option (c) is correct.

(d)

$$M-2I=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}-2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & 1 & 1\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}$$

$|M-2I|=0$; So, option (d) is incorrect.

13. **(a, d)** $-x+2y+5z=b_1$

$2x-4y+3z=b_2$

$x-2y+2z=b_3$

has at least one solution.

$$\therefore D=\begin{vmatrix}-1 & 2 & 5\\ 2 & -4 & 3\\ 1 & -2 & 2\end{vmatrix}=0 \text{ and } D_1=D_2=D_3=0$$

$$\Rightarrow D_1=\begin{vmatrix}b_1 & 2 & 5\\ b_2 & -4 & 3\\ b_3 & -2 & 2\end{vmatrix}$$

$=-2b_1-14b_2+26b_3=0\Rightarrow b_1+7b_2=13b_3$ .....(i)

(a) $D=\begin{vmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 1 & 2 & 6\end{vmatrix}=1(24-10)+1(10-12)$

$=14-2=12\neq 0$

Here, $D\neq 0\Rightarrow$ unique solution for any b_1, b_2, b_3.

(b) $D=\begin{vmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{vmatrix}$

$=1(-6+6)-1(-15+12)+3(-5+4)=0$

For atleast one solution

$D_1=D_2=D_3=0$

Now, $D_1=\begin{vmatrix}b_1 & 1 & 3\\ b_2 & 2 & 6\\ b_3 & -1 & -3\end{vmatrix}$

$=b_1(-6+6)-b_2(-3+3)+b_3(6-6)=0$

$D_2=\begin{vmatrix}1 & b_1 & 3\\ 5 & b_2 & 6\\ -2 & b_3 & -3\end{vmatrix}$

$= -b_1(-15+12) + b_2(-3+6) - b_3(6-15)$
$= 3b_1 + 3b_2 + 9b_3 = 0 \Rightarrow b_1 + b_2 + 3b_3 = 0$
not satisfies the Eq. (i)
It has no solution.

(c) $D = \begin{vmatrix} -1 & 2 & -5 \\ 2 & -4 & 10 \\ 1 & -2 & 5 \end{vmatrix}$

$= -1(-20+20) - 2(10-10) - 5(-4+4) = 0$
Here, $b_2 = -2b_1$ and $b_3 = -b_1$ satisfies the Eq. (i)
Planes are parallel.

(d) $D = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & -5 \end{vmatrix} = 1(0-12) - 2(-10-3) + 5(8-0) = 54$

$D \neq 0$
It has unique solution for any b_1, b_2, b_3.

14. **(b, c, d)**
$ax + 2y = \lambda$
$3x - 2y = \mu$

For unique solution, $\frac{a}{3} \neq \frac{2}{-2} \Rightarrow a \neq -3$

$\therefore$ (b) is correct.
For infinite many solutions and $a = -3$

$\frac{-3}{3} = \frac{2}{-2} = \frac{\lambda}{\mu} \Rightarrow \frac{\lambda}{\mu} = -1$ or $\lambda + \mu = 0$

$\therefore$ (c) is correct.

Also if $\lambda + \mu \neq 0$, then $\frac{-3}{3} = \frac{2}{-2} \neq \frac{\lambda}{\mu}$

$\Rightarrow$ system has no solution.
$\therefore$ (d) is correct.

15. **(a)** Given system of equations are:
$x + 2y + z = 7; x + \alpha z = 11; 2x - 3y + \beta z = \gamma$

$\therefore \Delta = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & \alpha \\ 2 & -3 & \beta \end{vmatrix} = 7\alpha - 2\beta - 3$

$\Delta_x = \begin{vmatrix} 7 & 2 & 1 \\ 11 & 0 & \alpha \\ \gamma & -3 & \beta \end{vmatrix} = 21\alpha - 22\beta + 2\alpha\gamma - 33$

$\Delta_y = \begin{vmatrix} 1 & 7 & 1 \\ 1 & 11 & \alpha \\ 2 & \gamma & \beta \end{vmatrix} = 4\beta + 14\alpha - \alpha\gamma + \gamma - 22$

$\Delta_z = \begin{vmatrix} 1 & 2 & 7 \\ 1 & 0 & 11 \\ 2 & -3 & \gamma \end{vmatrix} = 56 - 2\gamma$

(P) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma = 28$ then
$\Delta = 0, \Delta_x = \Delta_y = \Delta_z = 0$
So, Infinitely many solutions

(Q) If $\beta = \frac{1}{2}(7-3)$ and $\gamma \neq 28$ then
$\Delta = 0$ but $\Delta_z \neq 0$. So, no solution.

(R) If $\beta \neq \frac{1}{2}(7\alpha - 3), \alpha = 1$ and $\gamma \neq 28$ then
$\Delta \neq 0, \Delta_z \neq 0$ so, a unique solution

(S) If $\beta \neq \frac{1}{2}(7\alpha - 3), \alpha = 1$ and $\gamma = 28$ then
$\Delta \neq 0, \Delta_z \neq 0. \Delta_x = \Delta_y = 0$
So, unique solution and $x = 11, y = -2, z = 0$.

16. **(a)** Each element of set $\mathscr{A}$ is 3×3 symmetric matrix with five of its entries as 1 and four of its entries as 0, we can keep in diagonal either 2 zero and one 1 or no zero and three 1 so that the left over zeros and one's are even in number.
Therefore, taking 2 zeros and one 1 in diagonal the possible cases are $\frac{3!}{2!} \times \frac{3!}{2!} = 9$
and taking 3 ones in diagonal the possible cases are
$1 \times \frac{3!}{2!} = 3$
$\therefore$ Total number of elements of $\mathscr{A} = 9 + 3 = 12$

17. **(b)** Given system of linear equations : $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

This system will have unique solution if $|A| \neq 0$.
$\therefore$ The possible matrix A are

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$

$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

i.e., 6 possible matrices.

18. **(b)** For the given system to be inconsistent $|A| = 0$.
$\therefore$ The possible matrix A are

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (i), $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (ii), $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii)

and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (iv), $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (v), $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (vi)

On solving $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We find for $A =$ (i)
By Cramer's rule $D_1 = 0 = D_2 = D_3$
$\therefore$ infinite many solution
For $A =$ (ii)
By Cramer's rule $D_1 \neq 0$
$\Rightarrow$ no solution i.e. inconsistent.
Similarly we find the system as inconsistent in cases (iii), (v) and (vi).
Hence for four cases system is inconsistent.

19. **(a)** The given system of equations are
$x - 2y + 3z = -1$
$-x + y - 2z = k$
$x - 3y + 4z = 1$

Here $D = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$

and $D_2 = \begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & 1 & 4 \end{vmatrix} = k - 3 \neq 0$ if $k \neq 3$

$\therefore$ If $k \neq 3$, the system has no solutions.

Therefore, statement-1 is true and statement-2 is a correct explanation for statement - 1.

20. Since, AX = U has infinitely many solutions.

$\therefore |A| = 0 \Rightarrow \begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix} = 0$

$\Rightarrow a(bc - bd) + 1(d - c) = 0 \Rightarrow (d - c)(ab - 1) = 0$

$\therefore ab = 1$ or $d = c$

Again, $|A_3| = \begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix} = 0 \Rightarrow g = h$

$\Rightarrow |A_2| = \begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix} = 0 \Rightarrow g = h$

and $|A_1| = \begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix} = 0 \Rightarrow g = h$

$\therefore g = h, c = d$ and $ab = 1$ ...(i)

Now, $BX = V$

$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$ [From (i)]

[since, C_2 and C_3 are equal]

$\therefore BX = V$ has no solution.

$|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0$ [From (i)]

[$\because c = d$ and $g = h$]

$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df$ [$\because c = d$]

Since, $adf \neq 0 \Rightarrow |B_2| \neq 0$

$|B| = 0$ and $|B_2| \neq 0$

BX = V has no solution.

21. System of linear equations

$\lambda x + (\sin \alpha) y + (\cos \alpha) z = 0$

$x + (\cos \alpha) y + (\sin \alpha) z = 0$

$-x + (\sin \alpha) y - (\cos \alpha) z = 0$

has a non-trivial solution.

$\therefore \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$

$\Rightarrow \lambda(-\cos^2 \alpha - \sin^2\alpha) - \sin \alpha (-\cos \alpha + \sin \alpha)$
$+ \cos \alpha(\sin \alpha + \cos \alpha) = 0$

$\Rightarrow -\lambda + \sin \alpha \cos \alpha - \sin^2 \alpha + \sin \alpha \cos\alpha + \cos^2\alpha = 0$

$\Rightarrow \lambda = \cos^2\alpha - \sin^2\alpha + 2 \sin\alpha \cos \alpha \Rightarrow \lambda = \cos 2\alpha + \sin 2\alpha$

For $\lambda = 1$, $\cos 2\alpha + \sin 2\alpha = 1$

$\Rightarrow \frac{1}{\sqrt{2}} \cos 2\alpha + \frac{1}{\sqrt{2}} \sin 2\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos 2\alpha \cos \pi/4 + \sin 2\alpha \sin \pi/4 = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos (2\alpha - \pi/4) = \cos \pi/4 \Rightarrow 2\alpha - \pi/4 = 2n\pi \pm \pi/4$

$\Rightarrow 2\alpha = 2n\pi + \pi/4 + \pi/4;\ 2n\pi - \pi/4 + \pi/4$

$\therefore \alpha = n\pi + \pi/4$ or $n\pi$

22. The given system will have a non-trivial solution if

$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

On expanding along C_1, we get

$(28 - 21) \sin 3\theta - (-7 - 7) \cos 2\theta + 2(-3 - 4) = 0$

$\Rightarrow$ $7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$

$\Rightarrow$ $\sin 3\theta + 2 \cos 2\theta - 2 = 0$

$\Rightarrow$ $3 \sin\theta - 4 \sin^3\theta + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow$ $4 \sin^3\theta + 4 \sin^2\theta - 3 \sin \theta = 0$

$\Rightarrow$ $\sin \theta (2 \sin\theta - 1)(2 \sin \theta + 3) = 0$

$\therefore$ $\sin \theta = 0$ or $\sin \theta = 1/2$ [$\because \sin \theta \neq -3/2$]

$\Rightarrow$ $\theta = n\pi$ or $\theta = n\pi + (-1)^n \pi/6, n \in Z$.

23. For non-trival solution of the given system of equation,

$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$

$\Rightarrow 1(-4k + 6) - k(-12 + 4) + 3(9 - 2k) = 0 \Rightarrow k = \frac{33}{2}$

Substituting $k = \frac{33}{2}$ and putting $x = b$, where $b \in Q$, we get the system as

$33y + 6z = -2b$ ...(i)

$33y - 4z = -6b$ ...(ii)

$3y - 4z = -2b$ ...(iii)

On subtracting (ii) from (i), we get $10z = 4b \Rightarrow z = \frac{2}{5} b$

Now, from (i) $33y = -2b - \frac{12b}{5} = -\frac{22b}{5} \Rightarrow y = \frac{-2b}{15}$

$\therefore$ The solution is $x = b,\ y = \frac{-2b}{15},\ z = \frac{2b}{5}$